

GENUSES OF CLUSTER QUIVERS OF FINITE MUTATION TYPE

FANG LI, JICHUN LIU*, AND YICHAO YANG

ABSTRACT. In this paper, we study the distribution of the genres of cluster quivers of finite mutation type. First, we prove that in the 11 exceptional cases, the distribution of genres is 0 or 1. Next, we consider the relationship between the genus of an oriented surface and that of cluster quivers from this surface. It is verified that the genus of an oriented surface is an upper bound for the genres of cluster quivers from this surface. Furthermore, for any non-negative integer n and a closed oriented surface of genus n , we show that there always exist a set of punctures and a triangulation of this surface such that the corresponding cluster quiver from this triangulation is exactly of genus n .

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1. INTRODUCTION

Cluster quiver is a valuable notion in the theory of cluster algebras. Cluster algebras were introduced by Fomin and Zelevinsky in the famous paper [3]. Since then this subject has been studied extensively by many mathematicians. The original motivation was to give a combinatorial characterization of dual canonical bases in the theory of quantum groups, and for the study of total positivity for algebraic groups. Now cluster algebras are connected to various fields of mathematics such as representation theory, Poisson geometry, algebraic geometry, Lie theory, combinatorics and so on. One knows that cluster algebras are commutative algebras equipped with a distinguished set of generators, i.e., cluster variables.

Two types of cluster algebras are of important interests: finite type and finite mutation type. The former is a special case of the latter. Cluster algebras of finite type have been completely classified in [4] and skew-symmetric cluster algebras of finite mutation type have been completely

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*: Corresponding author

classified in [1]. The classification of cluster algebras of finite type is identical to the Cartan-Killing classification of semisimple Lie algebras and finite root systems. For a cluster algebra of finite type, there is a one-one correspondence between the set of cluster variables and that of almost positive roots (consisting of positive roots and negative simple roots). Additionally, the classification of skew-symmetric cluster algebras (equivalently, the classification of cluster quivers) of finite mutation type tells us that almost all skew-symmetric cluster algebras (equivalently, cluster quivers) of this type come from triangulations of surfaces except for 11 exceptional cases.

Given an oriented 2-dimensional Riemann surface S with boundary ∂S , let $M \subset S$ be a finite set of **marked points** such that each connected boundary component contains at least one such point. Marked points in the interior of S are called **punctures**. The pair (S, M) is simply called a **surface**. An **arc**([2]) is the homotopy class of a curve γ in S whose endpoints come from M such that

- γ does not intersect itself, except that its endpoints may coincide;
- except for the endpoints, γ is disjoint from M and ∂S ;
- γ does not cut out an unpunctured monogon or an unpunctured digon.

An **ideal triangulation** T is a maximal set of non-crossing (i.e., there are no intersections in the interior of S) arcs. For the details of the construction of cluster quivers from triangulations of surfaces, see section 2.2.

In this paper, all surfaces we consider are oriented surfaces; all subgraphs and subquivers are full.

In topological graph theory, the **genus** of a graph is the minimal genus of surfaces where the graph can be drawn without crossing. The **genus** of a quiver is defined to be that of its underlying graph. When discussing the genus of a quiver, one only needs to consider its simple underlying graph (without multiple edges and orientation). A graph (resp. quiver) is **planar** if it is of genus 0. It is well known that genus is a topological invariant for surfaces, as well as for topological graphs. A natural question is to find out the relation between the genus of a surface and that of a cluster quiver from this surface. As an answer, we have the main conclusion in this paper as follows:

Theorem 1.1. (i) *For a triangulation T of a surface S with genus g , let g' be the genus of the cluster quiver Q associated with T . Then, $g' \leq g$.*

(ii) *Furthermore, for any non-negative integer n and a closed oriented surface S_n of genus n , there exists a set of marked points M on S_n and an ideal triangulation P_n of S_n such that the corresponding cluster quiver T_n of P_n has genus n .*

From this result, we know that the genus of a surface is in fact an upper bound for the genres of cluster quivers from the triangulations of this surface and moreover any non-negative integer n can be reached as the genus of some cluster quiver from surface.

The paper is organized as follows. The requisite backgrounds on cluster quiver and its mutation, triangulation of surface are presented in § 2. In § 2.1, we give the basic definitions of matrix mutation and quiver mutation. We mention the fact that skew-symmetric matrices are in bijection with cluster quivers, also that matrix mutation and quiver mutation are compatible. In § 2.2, we recall some basic definitions and properties of triangulation of surface from [2]. One can see how to obtain a cluster quiver from a triangulation of surface and the compatibility between mutation of quiver and flip of triangulation. A cluster quiver comes from surface if and only if it is block-decomposable. As the end of this subsection, we restate the classification of skew-symmetric cluster algebras of finite mutation type.

§ 3 mainly deals with the genres of cluster quivers of finite mutation type. In § 3.1, we give the table of genus distribution of the 11 exceptional quivers by utilizing Keller's quiver mutation in Java [6]. In § 3.2, we first prove Theorem 1.1(i) which states that the genus of a surface is an upper bound of genres of cluster quivers obtained by triangulations of this surface. From this result, one can easily see that genus is a mutation invariant for cluster quivers from the surface of genus 0. As another application of this result, we give a sufficient condition for two quivers not to be mutation equivalent. The part (ii) of Theorem 1.1 is proved through constructing the graph R_n , using topological graph theory for genus n and the classification theorem of compact surfaces in algebraic topology.

2. PRELIMINARIES

2.1. Cluster quiver and its mutation. The notion of skew-symmetric matrix or equivalently of cluster quiver is crucial in the theory of cluster algebras. In the definition of cluster algebras, the most important ingredient is the so-called *seed mutation*. For our purpose in this paper, we only introduce *matrix mutation* (an important part of seed mutation) so as to understand the motivation of cluster quivers. For the details of the definitions of seed mutation and cluster algebra, we refer to [4].

Suppose $B = (b_{ij})$ is an $n \times n$ integer matrix. For $1 \leq k \leq n$, a **matrix mutation** μ_k at direction k transforms B into a new matrix $B' = (b'_{ij})$ where b'_{ij} is defined by the equation

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \frac{|b_{ik}b_{kj} + b_{ik}b_{kj}|}{2}, & \text{otherwise.} \end{cases}$$

Here, all matrices we consider are *skew-symmetric*. It is easy to see that matrix mutation transforms a skew-symmetric matrix into another one.

Given an $n \times n$ skew-symmetric matrix $B = (b_{ij})$, we can construct a quiver Q without loops and 2-cycles as follows: the vertex set is just the row/column indexes $1, 2, \dots, n$ of the matrix B and the number of arrows from i to j is defined to be b_{ij} if $b_{ij} > 0$.

Definition 2.1. A quiver without loops and 2-cycles is said to be a **cluster quiver**.

There is a one-one correspondence between the set of skew-symmetric matrices and that of cluster quivers. In fact, given a cluster quiver Q with n vertices, one can construct a skew-symmetric matrix $B = (b_{ij})$ defined by $b_{ij} = \# \{i \rightarrow j\} - \# \{j \rightarrow i\}$ where $\# \{i \rightarrow j\}$ denotes the number of arrows from i to j . According to this one-one correspondence, *quiver mutation* can be deduced from matrix mutation.

Definition 2.2. Suppose Q is a cluster quiver with n vertices where $Q_0 = \{1, 2, \dots, n\}$. For $k \in Q_0$, a **quiver mutation** μ_k at vertex k transforms Q into Q' where Q' is obtained by the following three steps:

- (1) For every path $i \rightarrow k \rightarrow j$, add a new arrow $i \rightarrow j$;
- (2) Reverse all arrows incident with k ;
- (3) Delete all 2-cycles.

One can easily see that the resulting quiver Q' is also a cluster quiver. Matrix mutation and quiver mutation are compatible in the following sense: given any $k \in 1, 2, \dots, n$, $\mu_k(Q_B) = Q_{\mu_k(B)}$ and $\mu_k(B_Q) = B_{\mu_k(Q)}$.

It is easy to verify that both matrix mutation and quiver mutation are involutions, i.e., $\mu_k^2 = 1$. If $Q' = \mu_{k_1} \mu_{k_2} \cdots \mu_{k_l}(Q)$ for some $k_1, k_2, \dots, k_l \in \{1, 2, \dots, n\}$, we will say Q and Q' are **mutation**

equivalent. Obviously it is an equivalence relation on the set of isomorphism classes of cluster quivers with n vertices. A cluster quiver(resp. skew-symmetric cluster algebra constructed from this quiver) is said to be of **finite mutation type** if the number of quivers in its mutation equivalent class is finite. Cluster quivers of this type have been completely classified in [1]. We will restate this classification theorem in §2.2.

2.2. Cluster quivers from surface. Given a surface (S, M) , the number of arcs in any triangulations of (S, M) is a constant. The following lemma gives the formula to calculate the number of arcs in a triangulation.

Lemma 2.3. [2] *For a triangulation of a surface, the following formula holds:*

$$(1) \quad n = 6g + 3b + 3p + c - 6$$

where n is the number of arcs; g is the genus of the surface; b is the number of connected boundary components; p is the number of punctures; c is the number of marked points on the boundary.

The arcs of an ideal triangulation cut the surface S into **ideal triangles**. The three sides of an ideal triangle do not have to be distinct, i.e., we allow **self-folded** triangle which can be shown as the following figure:

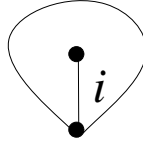


Figure 1

Given an ideal triangulation T , there is an associated signed adjacency matrix $B(T)$ (see [2] §4). Suppose the arcs in T are labeled by the numbers $1, 2, \dots, n$ and let the rows and columns of $B(T)$ be numbered from 1 to n . For an arc i , let $\pi_T(i)$ denote the arc defined as follows: if there is a self-folded ideal triangle in T folded along i (see Figure 1), then $\pi_T(i)$ is its remaining side; otherwise, we set $\pi_T(i) = i$.

For each non-self-folded triangle Δ , define the $n \times n$ integer matrix $B^\Delta = (b_{ij}^\Delta)$ by setting

$$b_{ij}^\Delta = \begin{cases} 1, & \text{if } \Delta \text{ has sides labeled } \pi_T(i) \text{ and } \pi_T(j) \text{ with } \pi_T(j) \text{ following } \pi_T(i) \text{ in the clockwise order;} \\ -1, & \text{if the same holds, with the counter-clockwise order;} \\ 0, & \text{otherwise.} \end{cases}$$

The matrix $B = B(T) = (b_{ij})$ is defined by

$$(2) \quad B = \sum_{\Delta} B^\Delta$$

where the sum is taken over all non-self-folded triangle Δ . It is easy to verify that $B(T)$ is skew-symmetric, and all its entries are equal to 0, 1, -1, 2 or -2. Therefore, given a triangulation T , we can firstly associate a skew-symmetric matrix $B(T)$ to T and then obtain a cluster quiver Q corresponding to $B(T)$, just as that given in §2.1. The corresponding cluster quiver Q_B of $B = B(T)$ is said to be **coming from surface**. Correspondingly, the cluster algebra defined by Q_B is also said to be *coming from surface*.

A **flip** is a transformation of an ideal triangulation T into a new triangulation T' obtained by replacing an arc γ with a unique different arc γ' and leaving other arcs unchanged. Flip of triangulation and mutation of matrix are compatible in the sense of the following proposition.

Proposition 2.4. ([2] Proposition 4.8) *Suppose that the triangulation \bar{T} is obtained from T by a flip replacing an arc k . Then $B(\bar{T}) = \mu_k(B(T))$.*

According to the Remark 4.2 in [2], all triangulations that we are interested in can be obtained by gluing together a number of puzzle pieces except for one case, i.e. the triangulation of 4-punctured sphere obtained by gluing three self-folded triangles to respective sides of an ordinary triangle, see Figure 2(I). There are three types of puzzle pieces as follows, see Figure 2(II).

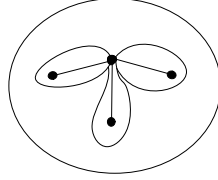


Figure 2(I)

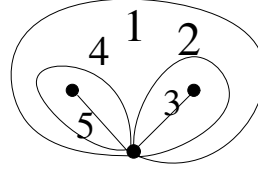
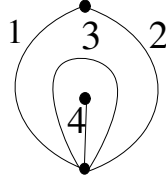
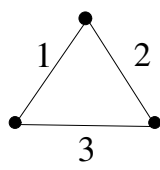


Figure 2(II)

These three types of puzzle pieces correspond to blocks of type I-V (see Figure 3) depending on whether the outer sides are lying on the boundary (for the details, see the proof of Theorem 13.3 in [2]). The vertices marked by unfilled circles in Figure 3 are called **outlets**.

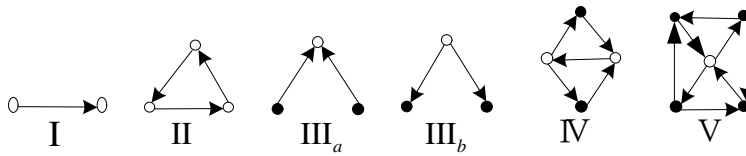


Figure 3

Definition 2.5. ([2]) *A quiver is said to be **block-decomposable** if it can be obtained from a collection of disjoint blocks by the following procedure:*

- (1) *take a partial matching of the combined set of outlets (matching an outlet to itself or to another outlet from the same block is not allowed);*
- (2) *glue the outlets in each pair of matching;*
- (3) *remove all 2-cycles.*

According to Theorem 13.3 in [2], a cluster quiver is coming from surface if and only if it is block-decomposable.

The following theorem gives a complete classification of skew-symmetric cluster algebras of finite mutation type.

Lemma 2.6. [1]. A skew-symmetric cluster algebra \mathcal{A} of rank n is of finite mutation type if and only if:

- (1) \mathcal{A} is coming from surface ($n \geq 3$);
- or (2) $n \leq 2$;
- or (3) \mathcal{A} is one of the 11 exceptional types (see Figure 4):

$$E_6, E_7, E_8, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}, X_6, X_7$$

that is, \mathcal{A} has a cluster quiver at one of the above types.

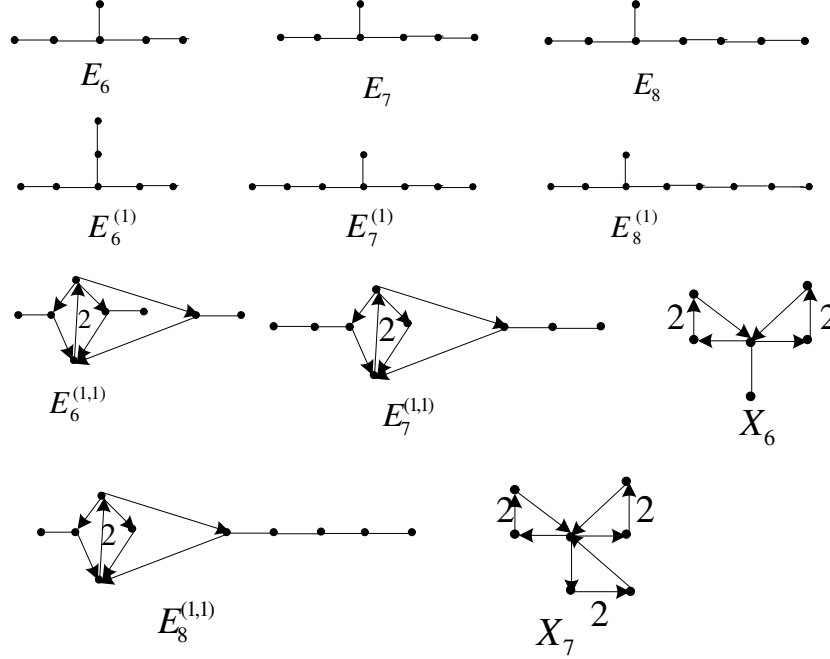


Figure 4

3. DISTRIBUTION OF GENUSES OF CLUSTER QUIVERS OF FINITE MUTATION TYPE

3.1. Genuses of exceptional cluster quivers. In this section, we give the table of distribution of genres of the 11 exceptional cluster quivers in the classification of finite mutation type. Our main tool is Keller's quiver mutation in java([6]). To obtain the following table, one should note the following facts:

(1) $E_6, E_7, E_8, E_6^{(1)}, E_7^{(1)}$ and $E_8^{(1)}$ are trees. According to Lemma 1.1 in [10], any orientations on the same tree are mutation equivalent.

(2) E_6, E_7, E_8 and $E_8^{(1)}$ are full subgraphs of the underlying graph of $E_8^{(1,1)}$; $E_6^{(1)}$ is a full subgraph of the underlying graph of $E_6^{(1,1)}$; $E_7^{(1)}$ is a full subgraph of the underlying graph of $E_7^{(1,1)}$. Since any quiver mutation equivalent to a full subquiver of Q must be a full subquiver of some Q' which is mutation equivalent to Q , we first test the mutation classes of $E_6^{(1,1)}, E_7^{(1,1)}$ and $E_8^{(1,1)}$ in order to see their genus distribution.

(3) To see the genus of a quiver, we only need to see its underlying graph. Hence when doing the quiver mutation in java due to [6], we can choose the mutation class under graph isomorphism. This can cut down the number of quivers in mutation class greatly which we have to consider.

(4) We check all the quivers in the mutation classes of $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ and find that all of them are planar. So are the other exceptional cluster quivers of E type.

In the following distribution table, the *total number* means the number of quivers in the mutation class up to quiver isomorphism, and the *number of genus 0* (resp. 1) means the number of quivers (up to quiver isomorphism) in the mutation class whose genus is 0 (resp. 1).

11 exceptional cluster quivers	total number	the number of genus 0	the number of genus 1
E_6	21	21	0
E_7	112	112	0
E_8	391	391	0
$E_6^{(1)}$	52	52	0
$E_7^{(1)}$	338	338	0
$E_8^{(1)}$	1935	1935	0
$E_6^{(1,1)}$	27	27	0
$E_7^{(1,1)}$	217	217	0
$E_8^{(1,1)}$	1886	1886	0
X_6	4	1	3
X_7	2	1	1

From the above table, one can easily see that the genus of the quiver of type E is invariant under quiver mutation; but the genus of the quiver of type X will vary under quiver mutation.

Proposition 3.1. *There are exactly 4 non-planar cluster quivers of exceptional finite mutation types, with genres 1, which are listed as follows, where the quivers (1), (2), (3) are in the*

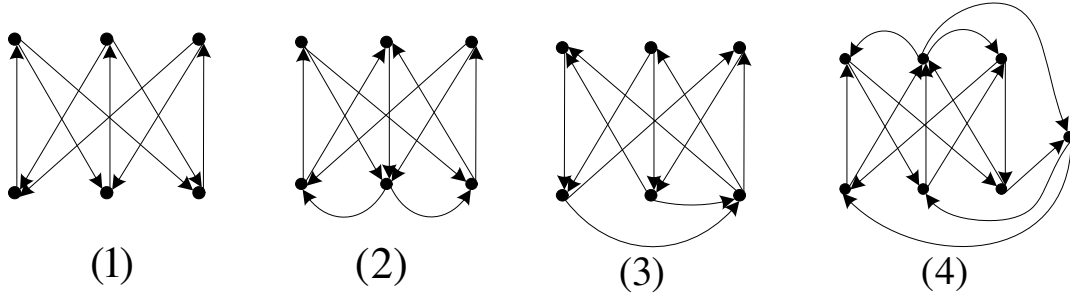


Figure 5

mutation-equivalent class of X_6 , the quiver (4) is in the mutation-equivalent class of X_7 .

Proof. For the convenience of describing the mutations at X_6 and X_7 to obtain these four quivers, we will label the vertices of X_6 and X_7 as Figure 6. Then, we can get the four quivers in Figure 5

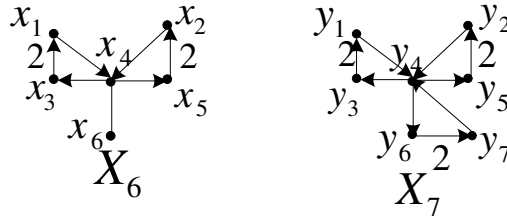


Figure 6

as follows:

- the quiver (1) can be obtained from X_6 by mutation on the vertices x_4 and x_6 ;
- the quiver (2) can be obtained from X_6 by mutation on the vertex x_4 ;
- the quiver (3) can be obtained from X_6 by mutation on the vertices x_4 and x_3 ;
- the quiver (4) can be obtained from X_7 by mutation on the vertex y_4 .

□

3.2. Proof of the main conclusion. We will begin by proving the first part of the theorem, i.e., the genres of cluster quivers obtained from the triangulations of a surface are not greater than that of the surface.

Proof of Theorem 1.1(i). By the correspondence of puzzle pieces and blocks, each puzzle piece corresponds to a block of types I-V. For each puzzle piece, we put its corresponding block into the face bounded by it. If two puzzle pieces have a common edge, then we glue two vertices corresponding to the common edge between these two blocks. Hence we obtain the quiver Q of T in this way and moreover the underlying graph of Q can be drawn without self-crossing on the surface S . We then have $g' \leq g$ by the definition of genus of quiver.

To complete the proof of the theorem, we should consider the only exceptional case the triangulation of which cannot be obtained by gluing the puzzle pieces. Let T be the triangulation of 4-punctured sphere obtained by gluing three self-folded triangles to respective sides of an ordinary triangle. The corresponding cluster quiver of T can be obtained by gluing four blocks of type II, which can be shown as the following Figure 7. In this figure, for $i = 1, 2$ and 3 , i and i' denote the corresponding vertices of two arcs in the same self-folded triangles. Obviously it is a planar quiver and hence in this case $g' = g = 0$.

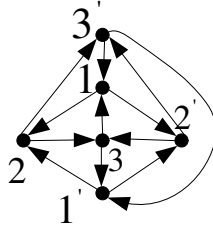


Figure 7

This completes the proof.

□

In order to prove Theorem 1.1(ii), we need some preliminaries from [5] and [7].

Let us firstly introduce a class of graphs with arbitrary large genus. For each positive integer n , the graph R_n is constructed as follows (see [5] Example 3.4.2):

There are $n + 1$ concentric cycles which consists of $4n$ edges. Additionally, there are $4n^2$ inner edges connecting the $n + 1$ cycles to each other and $2n$ outer edges adjoining antipodal vertices on the outermost cycle.

The following Figure 8 gives the example R_2 for such graph in the case $n = 2$.

It was shown in [5] that R_n is of genus n .

Secondly, we recall some basic notions and facts in [7].

The classification theorem of compact(or closed) surfaces (see Theorem 5.1 of [7] Chapter 1) asserts that any compact surface could be either homeomorphic to a sphere, or to a connected sum of tori, or to a connected sum of projective planes. The compact surfaces can be considered as the quotient space of a polygon with the directed edges identified in pairs. There is a convenient way to indicate which paired edges are to be identified in such a polygon. We give a letter

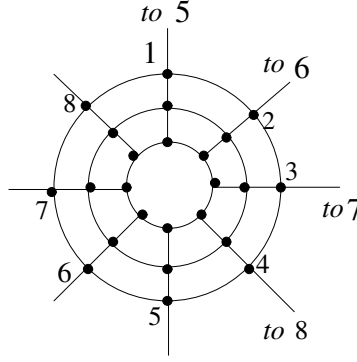


Figure 8

(for example a, b, c, \dots) to each paired edges such that different pairs receive different letters. Starting at a definite vertex, we traverse the boundary of the polygon in either a clockwise or counter clockwise fashion. If the arrow on an edge points in the same traversing direction, then we will put no exponent (or the exponent $+1$) on the letter for that edge; otherwise, we will write the letter for that edge with the exponent -1 . For example, the identifications of the following hexagon in Figure 9 can be indicated by the symbols as $a_1 a_1 a_2 a_2^{-1} a_3 a_3^{-1}$.

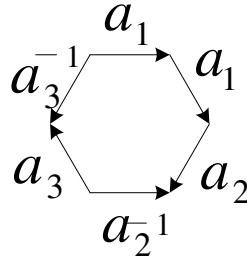


Figure 9

The symbols corresponding to the surfaces mentioned in the classification theorem are as follows (see [7] §5):

- (1) The sphere: aa^{-1} .
- (2) The connected sum of n tori:

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1}.$$

- (3) The connected sum of n projective planes:

$$a_1 a_1 a_2 a_2 \cdots a_n a_n.$$

Given a polygon, if the letter designating a certain pair of edges occurs with both exponents $+1$ and -1 in the symbol, then this pair of edges is said to be the **first kind**; otherwise the pair is said to be the **second kind** ([7]). From the proof of Theorem 5.1 in [7], we know that if all the pairs of edges are of the first kind, then the resulting surface is oriented; if there exists a second kind pair, then the resulting surface is non-oriented. Moreover, since the first kind pair of adjacent edges can be eliminated, the resulting surface of a $4n$ -gon with pairs all of the first kind is an oriented surface with genus at most n .

For the preparation of the proof of Theorem 1.1(ii), we first prove the following lemma.

Lemma 3.2. *For an arbitrary non-negative integer n , there always exists a block-decomposable cluster quiver T_n such that the genus $g(T_n)$ of T_n satisfies that $g(T_n) \geq n$.*

Proof. Given a graph R_n as above, label the $n + 1$ cycles from innermost to outermost by 1 to $n + 1$. For each $i \in \{1, 2, \dots, n\}$, there are $4n$ rectangles between the i th cycle and the $(i + 1)$ th cycle. For the outermost cycle, there exist $2n$ rectangles between the $(n + 1)$ th cycle and itself. Two rectangles are said to be **neighbored** if they share a common edge; otherwise, they are said to be **distant**. It is easy to observe that there are $4n^2 + 2n$ rectangles in R_n . Given any rectangle A in R_n , we first choose four rectangles distant to A but having a common vertex with A ; for these four rectangles, we do the same thing as previous step; continuing this process, we will obtain a maximal set of mutually distant rectangles. This is denoted by \mathcal{S} . This set contains $2n^2 + n$ rectangles. The other $2n^2 + n$ rectangles form another maximal set of mutually distant rectangles. This is denoted by \mathcal{T} . Trivially, these two sets \mathcal{S} and \mathcal{T} are independent of the choice of the original rectangle A . Consider the set \mathcal{S} , each rectangle in \mathcal{S} can be obtained by gluing four blocks of type II as the following Figure 10:

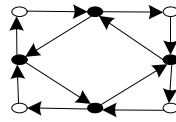


Figure 10

For the innermost cycle, there are $2n$ edges which do not lie in any rectangles of \mathcal{S} . We can then use one block of type IV to substitute each such edge. For all these $2n$ edges, we need $2n$ blocks of type IV.

In summary, we obtain a quiver T_n by gluing $8n^2 + 4n$ blocks of type II and $2n$ blocks of type IV. According to the construction of T_n , obviously, R_n is a subgraph of the underlying graph of T_n . Therefore, the genus $g(T_n) \geq g(R_n) = n$. The quiver of the following Figure 11 is the example of T_n in the case $n = 2$, where the vertices labeled by the same numbers should be glued together. \square

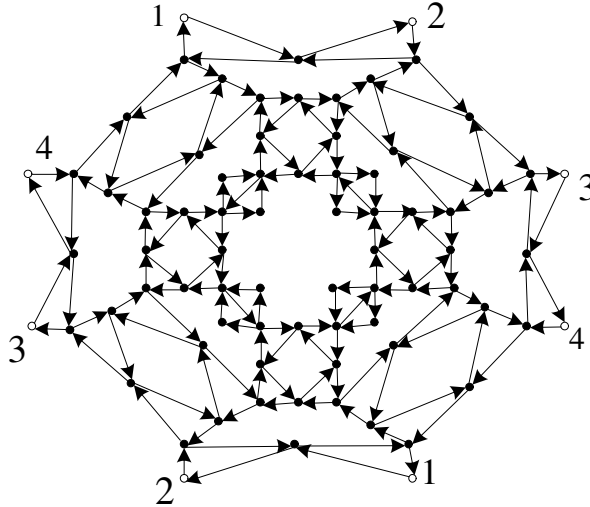


Figure 11

Now, we can give the proof of part (ii) of Theorem 1.1. The proof will be based on the fact that the quiver T_n given in the proof of Lemma 3.2 can be obtained from a closed surface of genus n .

Proof of Theorem 1.1(ii). By Lemma 3.2, $g(T_n) \geq n$. It is easy to check that T_n is a uniquely block-decomposable quiver and hence T_n can be uniquely encoded by its corresponding triangulation, that is, blocks of type II are encoded by puzzle pieces of first type (see the left graph in Figure 2(II)) and blocks of type IV are encoded by puzzle pieces of second type (see the middle graph in Figure 2(II)). In order to draw T_n , we firstly draw a planar quiver T'_n which has $4n$ unglued outlets. After gluing these $4n$ outlets in pairs, one obtains T_n , where each pair consists of one outlet and its opposite one. As example in the case $n = 2$, see Figure 11. Now, we will construct a closed surface S_n of genus n and a triangulation P_n of S_n such that its corresponding cluster quiver is T_n .

We will chase T_n from innermost to outermost. Blocks of type II and type IV are encoded by puzzle pieces of first type and second type respectively. For the outermost $4n$ oriented triangles in T'_n , we let each of them corresponds to a puzzle piece of first type. Thus we can first obtain a $4n$ -gon with triangulation. Denote this $4n$ -gon with triangulation as S'_n . Then we can obtain a closed oriented surface S_n by identifying the edges of S'_n in pairs and gluing all outermost vertices into one, and then obtain a triangulation P_n of S_n such that its corresponding quiver is exactly T_n . For the case T_2 , its corresponding S'_2 can be shown as the following Figure 12. To obtain S_2 and P_2 , one only need to glue the edges labeled by the same number in pairs and to glue all outermost 8 vertices into one.

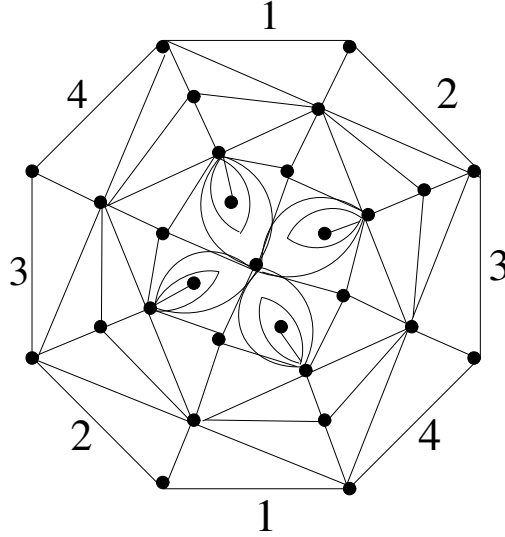


Figure 12

By the proof of the classification theorem of compact surfaces in [7], the genus of S_n is at most n .

Since T_n is obtained from a triangulation of S_n , by Theorem 1.1(i), $g(T_n) \leq n$.

On the other hand, by Lemma 3.2, $g(T_n) \geq n$. Hence, $g(T_n) = n$.

For the genus $g(S_n)$ of S_n , since $n = g(T_n) \leq g(S_n) \leq n$, we also have $g(S_n) = n$.

Then Theorem 1.1(ii) easily follows from the fact that all closed oriented surfaces with the same genus are homeomorphic. \square

3.3. Applications and further problems. As an application of Theorem 1.1(i), we firstly give two corollaries.

Corollary 3.3. *Let S be a surface of genus 0 and M be a set of marked points of S . Given any triangulation T of (S, M) , suppose Q is the associated cluster quiver from T , then all quivers in the mutation-equivalent class of Q are of genus 0.*

Besides the cluster quivers of type E in Section 3.1, this corollary gives another class of cluster quivers of finite mutation type whose genres are invariant under mutation.

Corollary 3.4. *Let S be a surface of genus g with M the set of marked points. For any triangulation T of (S, M) , let Q be its corresponding quiver and let Q' be another cluster quiver of genus g' such that $g' > g$, then Q and Q' are not mutation-equivalent.*

Proof. According to Proposition 12.3 in [2], all quivers in the mutation-equivalent class of Q are the corresponding quivers of some triangulations of (S, M) . Hence, by Theorem 1.1(i), the genres of these quivers are not greater than g . Hence Q' is not in the mutation-equivalent class of Q , that is, Q and Q' are not mutation-equivalent. \square

This corollary gives us a necessary condition for two quivers, one is from triangulation of a surface and the other is non-planar, with the same numbers of vertices, to be mutation-equivalent.

Remark 3.5. *An easy calculation shows that the number of marked points on the closed surface S_n in the proof of Theorem 1.1(ii) is $4n^2 + 2n + 2$. For example in the case $n = 2$, one can easily see that there are 22 marked points on S_2 , here the outermost 8 marked points in S'_2 (see Figure 12) are glued into one.*

Theorem 1.1(ii) tells us that given a closed surface S of genus n , the upper bound of genres of quivers from triangulations of S given in part (i) of Theorem 1.1 can be reached.

On the other hand, the lower bound 0 of genres can also be reached, that is, given any closed oriented surface S with genus n , there always exists a triangulation T of S such that the corresponding cluster quiver Q of T is planar.

In fact, if the closed surface is a sphere, it obviously holds by Corollary 3.3. If the closed surface S is of genus $n (\geq 1)$, then it is homeomorphic to the connected sum of n -tori. In this case, the symbol of its corresponding polygon is $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1}$. A triangulation T of S with two punctures is as follows in Figure 13. For this triangulation, the outer $4n$ vertices in fact come from the same puncture and the only inner vertex is the other puncture. One can easily check that the corresponding cluster quiver Q of T is planar.

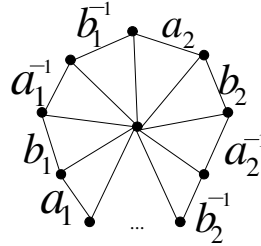


Figure 13

Restricting the discussion to torus, we can reach the following conclusion:

Proposition 3.6. *For a given cluster quiver Q from the torus S with p punctures, there exists at least one planar quiver in the mutation-equivalent class of Q .*

Proof. According to Proposition 12.3 in [2], the corresponding quivers from all triangulations of S are mutually mutation-equivalent. Hence, we only need to find a triangulation T of S such that its corresponding quiver is planar.

For the convenience of describing the desired triangulation, we first restate how a torus is constructed. Given two circles C and C' , assume the radius of C is greater than that of C' . Let the center of C' run along C for one round, then a torus is built. The circle C is called a **basic circle** for this torus.

For the torus S with p punctures, we construct a triangulation T as follows:

For each puncture, construct a closed arc on S perpendicular to the basic circle such that its two endpoints are both coincided at this puncture. Thus, we have such p arcs. These p arcs cut down the torus into p pieces of cylinders. For each cylinder, drawing an arc between two punctures, we obtain a rectangle. Moreover, we draw a diagonal in this rectangle. The corresponding quiver from such a rectangle with diagonal is shown as in the following Figure 14.

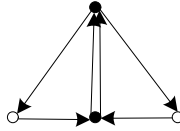


Figure 14

All such p rectangles with diagonal are arranged continuously together to form a graph. The quiver Q of T is obtained by gluing p pieces of such quivers along the outlets. Obviously, it is a planar quiver. For example, in the case $p = 3$, the triangulation can be shown as in Figure 15(1), where the numbers $1, \dots, 9$ label the arcs, and its corresponding cluster quiver is shown as in Figure 15(2).

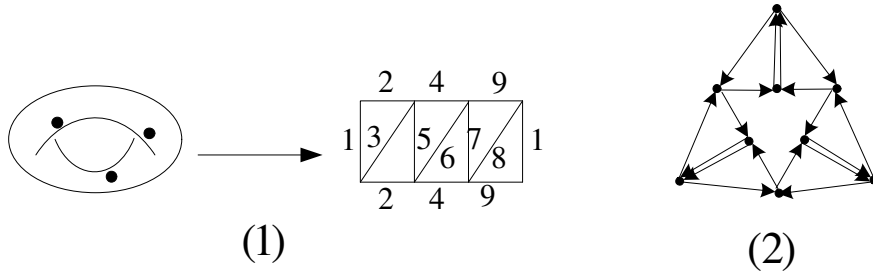


Figure 15

□

Since both of the upper bound and lower bound can reach for genres of cluster quivers from closed surface, based on Theorem 1.1 and Proposition 3.6, we propose the further interesting problems as follows:

Problem 3.7. For any closed surface S with genus n and $0 \leq i \leq n$, does there exist a certain number of punctures and an ideal triangulation $T^{(i)}$ of S such that the corresponding cluster quiver Q_i from $T^{(i)}$ is of genus i ?

Problem 3.8. Given a closed surface S of genus n . Find out the minimal number of punctures on S with the property that there exists an ideal triangulation T of S such that the corresponding cluster quiver Q_n of T is exactly of genus n .

For the case of torus, we know at least one planar quiver in each mutation-equivalent class according to Proposition 3.6. Hence for a given number of punctures we can check the corresponding mutation-equivalent class of this planar quiver by Keller's quiver mutation in Java [6]. Since the genus of a quiver has nothing to do with the orientations of arrows, we can choose the mutation-equivalent class under graph isomorphism when doing quiver mutation in Java.

For the cases $p = 1$ and $p = 2$, all quivers in their two mutation-equivalent classes are planar. When $p = 3$, there exists exactly one quiver of genus 1 (see Figure 16) in the mutation class. Therefore, the answer of Problem 3.8 for the case of torus is $p = 3$, which is much smaller than the number $4 \times 1^2 + 2 \times 1 + 2 = 8$ of punctures given in Remark 3.5 when constructing T_1 from torus.

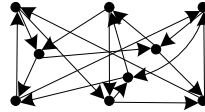


Figure 16

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DEPARTMENT OF MATHEMATICS, ZHEJIANG UNIVERSITY, HANGZHOU 310027, CHINA
E-mail address: fangli@zju.edu.cn

DEPARTMENT OF MATHEMATICS, ZHEJIANG UNIVERSITY, HANGZHOU 310027, CHINA
E-mail address: liujc1982@126.com

DEPARTMENT OF MATHEMATICS, ZHEJIANG UNIVERSITY, HANGZHOU 310027, CHINA
E-mail address: yyc880113@163.com